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CONTINUOUS AND DISCRETE MODELS OF BOUNDED ONE-DIMENSIONAL MEDIA IN THE THEORY OF VISCOELASTICITY†

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The relation between the discrete and continuous models of a one-dimensional viscoelastic medium is discussed. Beginning with the discrete model, for the linear case it is shown how to construct a series of partial differential equations which might be thought of as intermediate between the differential-difference equation for a chain of discrete masses and the equation for a continuous medium. For these intermediate equations (one of which will be, in particular, the equation of a vibrating continuous medium) we give the conditions under which one is justified in replacing the initial continuous model by a discrete chain. For the non-linear case, the analogue of the Fermi–Pasta–Ulam (FPU) problem with strongly non-linear constraints is considered. This version of the FPU problem can be applied to a chain with non-linear viscoelasticity.

1. CONTINUOUS MODELS

It is usual to study the longitudinal vibrations of one-dimensional systems with relaxation and creep using a continuous model, that is, by studying the vibrations of the corresponding continuous medium, described by the equation

$$\rho u_{tt} = \sigma_x - \eta u_t + f, \quad \eta = \text{const} > 0 \tag{1.1}$$

and a certain constitutive relation (a relation between the unknown functions ε and σ). Here $\varepsilon(x, t) = u_x(x, t)$, and $u(x, t)$ and $\sigma(x, t)$ are, respectively, the longitudinal displacement and stress of the one-dimensional continuous medium in section x at time t; $\rho = \text{const} > 0$ and $f(x, t)$ are, respectively, the linear mass densities and applied distributed load.

Linear constitutive relations are frequently used to illustrate discrete mechanical models made up of combinations of Hooke's, Newton's and other elements [1–3]. For instance, for a "standard" viscoelastic material, the constitutive relation can be put in the form [4]

$$L_0\sigma + L_1\sigma_t = b_0\varepsilon + b_1\varepsilon_t$$

where L_0, L_1, b_0, b_1 are certain constants and $\varepsilon = u_x$.

It is quite common in the linear theory of viscoelasticity in the general case to consider a constitutive relation of the form [1]

$$\sum_{r=0}^{R_1} L_r \frac{\partial^r}{\partial t^r}\sigma = \sum_{r=0}^{R_2} b_r \frac{\partial^r}{\partial t^r}\varepsilon \tag{1.2}$$

where L_r, b_r are certain constant coefficients. In that case, eliminating the function σ from (1.1) and (1.2), the equation obtained for the function u is [1]

$$\sum_{r=0}^{R_1} L_r \frac{\partial^r}{\partial t^r}(\rho u_{tt} + \eta u_t - f) = \sum_{r=0}^{R_2} b_r \frac{\partial^{r+2}}{\partial t^r \partial x^2} u \tag{1.3}$$

For bounded media, boundary conditions for Eq. (1.3) can be assigned in the form

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$$u(0,t) = u(l,t) = 0, \quad l > 0 \tag{1.4}$$

for example.

As far as the initial conditions are concerned, the constitutive relation (1.3) implies that the particular continuous medium possesses a memory [3, 4]. Thus, to be able to construct a unique solution, in addition to the usual initial conditions for the function $(x, t) \to u(x, t)$

$$u(x,0) = \beta_0(x), \quad u_t(x,0) = \beta_1(x) \tag{1.5}$$

where $x \to \beta_0(x), x \to \beta_1(x)$ are given functions, it is also necessary to describe the evolution of the medium for $t < 0$, or assign its initial state, for example, by indicating the functions

$$x \to \sigma(x,0),\ldots, x \to \partial^{R_1-1}\sigma(x,0)/\partial t^{R_1-1} \tag{1.6}$$

If we know the functions (1.5) and (1.6), we can uniquely determine all the necessary initial conditions for the function $(x, t) \to u(x, t)$ from the pair of equations (1.1), (1.2)

$$u(x,0) = \beta_0(x),\ldots,\frac{\partial^{R-1}}{\partial t^{R-1}}u(x,0) = \beta_{R-1}(x), \quad R = \max\{R_1+2, R_2\} \tag{1.7}$$

(In mechanics, as a rule, either $R_1 = R_2$ or $R_1 = R_2 - 1$ [1].)

2. DISCRETE MODELS

The study of the vibrations of a system described by a mechanical model in the form of a discrete chain of physical points joined by links with relaxation and creep involves an analysis of a system of two differential-difference equations (DDE), comprising the equation

$$m\ddot{y} = s_{j+1} - s_j - \xi\dot{y}_j + F_j, \quad \xi = \text{const} > 0 \tag{2.1}$$

(m is the mass of point j, F_j is the external longitudinal force applied to point j and s_j is the force with which point j acts on point $(j-1)$), and the equation

$$\sum_{r=0}^{R_1} L_r \frac{d^r}{dt^r} s_j = \sum_{r=0}^{R_2} B_r \frac{d^r}{dt^r}\Delta_j, \quad \Delta_j = y_j - y_{j-1} \tag{2.2}$$

which is used as the constitutive relation. Here L_r, B_r are certain constants.

Eliminating s_j, say, from (2.1) and (2.2), we obtain the differential-difference equation ("chain")

$$\sum_{r=0}^{R_1} L_r \frac{d^r}{dt^r}(m\ddot{y}_j + \xi\dot{y}_j - F_j) = \sum_{r=0}^{R_2} B_r \frac{d^r}{dt^r}(y_{j+1} - 2y_j - y_{j-1}) \tag{2.3}$$

Remark. Equations which contain the operations of differentiation and taking the difference, each with respect to the same argument, are sometimes referred to as differential-difference equations (see [5], for example). However, following the classical tradition [6], the term differential-difference equation will be used here for equations in which the operations of differentiation and taking the difference are with respect to different arguments.

For Eq. (2.3) we can also set boundary conditions of the form

$$y_0(t) = y_N(t) = 0 \tag{2.4}$$

for example, and initial conditions similar to conditions (1.7)

$$y_j(0) = \alpha_0(j),\ldots, y_j^{(R-1)}(0) = \alpha_{R-1}(j), \quad j = 1,\ldots,N-1 \tag{2.5}$$

The sets $\{\alpha_0(j)\}, \ldots, \{\alpha_{R-1}(j)\}$ are assumed given.

3. THE RELATION BETWEEN THE CONTINUOUS AND DISCRETE MODELS IN THE LINEAR CASE

Problem (2.3), (2.4) and (2.5) can be thought of as a possible scheme of the method of straight lines [7, 8] for problem (1.3), (1.4), (1.7)

$$m = \rho h, \quad B_r = b_r/h, \quad \xi = \eta h, \quad N = lh \tag{3.1}$$

(the parameter $h > 0$ acts as the distance between physical points in a state of static equilibrium) and the functions $(x, t) \to f(x, t), x \to \beta_r(x)$ $(r = 0, \ldots, R-1)$ are associated in one way or another with the functions $t \to F_j(t)$ and sets $\{\alpha_r(j)\}, r = 0, \ldots, R-1$. One possible question is whether the solution of that system converges to the solution of the problem for Eq. (2.3).

For certain problems, as in the theory of viscoelasticity, it seems more natural to start with the continuous model, but for others, as in solid-state physics or for the generalized Zhukovskii problem [9], it is better to start with the discrete model. However, it is impossible on mathematical grounds to decide from which of the two models, discrete or continuous, to start and which should be taken as its approximation. In that sense, the two models can be regarded as equivalent.

The obvious question that arises is in some ways the dual of the question of the convergence of the method of straight lines: if we start with a discrete chain, how well does the continuous model reflect the properties of its solutions?

Also, in the method of straight lines one could attempt to improve the accuracy of the approximation of the functions $y_j(t)$ to the solution $u(x, t)$ for $x = jh$ by increasing the number of points N and thereby reducing the value of h, while preserving the quantities ρ, L_r, b_r, η and the functions $(x, t) \to f(x, t)$. However, if the initial discrete chain is fixed (that is, the quantities $x \to \beta_r(x)$ are fixed and the functions N, h, m, L_r, B_r, ξ and sets $t \to F_j(t)$ are given), how can one reduce the quantity $|y_j(t) - u(jh, t)|$?

We know from the theory of difference schemes (see [10], for example) that various schemes of the method of straight lines can be used to approximate a given partial differential equation. Equation (2.3), regarded as one possible scheme, is typical insofar as it has an immediate physical interpretation.

However, there is no unique partial differential equation whose solutions will approximate the solution of the original equation even in the case of the DDE (2.3). Naturally, not every partial differential equation of this kind will have a direct physical interpretation. In the non-linear case, we have the example of the Boussinesq equations (relative to the function u_x) considered in [11].

$$\rho u_{tt} = E_1\left(u_{xx} + \frac{h^2}{12} u_{xxxx}\right) + 3bh^2 u_x^2 u_{xx} \tag{3.2}$$

and Eq. (1.1) (with $\eta = 0, f(x, t) = 0$) with the constitutive

$$\sigma = E_1\varepsilon + b|\varepsilon|^p \operatorname{sgn}(\varepsilon) \quad (E_1 \geqslant 0, \quad b > 0, \quad p = 3) \tag{3.3}$$

considered in [12]. These equations correspond to continuous models for Eq. (2.1) (with $\xi = 0, F_j(t) = 0$) with the constitutive relation

$$s_j = c_1\Delta_j + B|\Delta_j|^p \operatorname{sgn}(\Delta_j) \quad (c_1 > 0, \quad B > 0, \quad p = 3) \tag{3.4}$$

(the problem of finding periodic solutions of Eq. (2.1) with a non-linear constitutive relation is known as the Fermi–Pasa–Ulam problem [13]).

4. CONTINUOUS MODELS OF DISCRETE CHAINS

Since the question of how to construct difference schemes of the method of straight lines for partial differential equations has been studied thoroughly, we will concentrate slightly more on various methods of constructing continuous analogues of discrete chains.

One of the most common methods of constructing continuous analogues is the following. The discrete variable $j \in \mathbb{Z}$ is formally replaced by the continuous variable $x \in \mathbb{R}^1$, and the function $(j, t) \to y_j(t)$ is replaced by the function $(x, t) \to Z(x, t)$, such that $Z(jh, t) = y_j(t)$, where the constant $h > 0$ is taken as the distance between adjacent points in a state of equilibrium. If the function $(x, t) \to Z(x, t)$ is analytic with respect to x, at each point $x = jh$ it can be expanded in terms of the variable x and substituted into the equation of the chain. Confining ourselves to terms of order with respect to h not higher than some

$d > 0$, we obtain the partial differential equation which is taken as the continuous analogue of the chain. Equation (3.2) of [11] was obtained in this way.

We will now go through this method of obtaining a continuous analogue in detail, using the example of chain (2.3). We introduce the function $(x, t) \to Z(x, t)$ such that $Z(jh, t) = y_j(t)$. Formal expansion at each point $x = jh$ yields the equation

$$m\ddot{Z}(jh,t)+\xi Z(jh,t)=c\sum_{n=1}^{\infty}\frac{2h^{2n}}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}Z(jh,t)+F_j(t) \tag{4.1}$$

Confining ourselves to terms on the right-hand side of Eq. (4.1) which are of order with respect to h not higher than some fixed $d \geqslant 0$, and replacing the function $t \to F_j(t)$ by some function $(x, t) \to hf(x, t)$, we obtain the approximate continuous model of chain (2.3) in the form

$$mu_{tt}(x,t)+\xi u_t(x,t)=c\sum_{n=1}^{M}\frac{2h^{2n}}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}u(x,t)+hf(x,t) \tag{4.2}$$

where $2M = d$.

Definition. An equation of type (4.2), obtained by discarding terms of order higher than d with respect to h is called an intermediate continuous model of the original chain of order d. In particular, putting $d = 2$ in (4.2), we obtain the usual wave equation, and putting $d = \infty$, we formally obtain a differential-difference equation for a chain.

The difference between Eq. (4.1) and Eq. (4.2) even when $d = \infty$ is worth noting: (4.2) holds for all $x \in [0, l]$, whereas (4.1) holds only for a finite number of values $x = jh \in [0, l], j = 0, \dots, N$.

Thus, generally speaking, neither the extension of values of the functions $t \to F_j(t)$ to all $x \in [0, l]$ (that is, the choice of the function $(x, t) \to f(x, t)$) nor the condition that should be satisfied by the equation instead of (4.1) when $x \neq jh$ is unique. It is usually assumed that the same equation is satisfied for $x \neq jh$ also, but in principle there are other possibilities.

It is widely thought that if the number of physical points is large and the distances between them are small, then this system moves in a similar way to the corresponding continuous medium (see [14–16], for example). This was believed even at the time of the famous dispute about the vibrations of a string between D'Alembert, Euler, Bernoulli and Lagrange.

However, it is assumed right from the start when deriving Eq. (1.1) (unlike Eq. (4.1)) that it is satisfied for all $x \in [0, l]$ (this follows from the actual representation of the continuous medium). This is one obvious explanation for a possible fundamental difference in the nature of the solutions for a chain and a continuous medium (see [17, 18], for example).

Two problems arise when considering how close the solutions of the discrete and continuous models are.

Problem 1. For a given fixed discrete chain, is there a continuous model for which the quantity $|y_j(t) - u(jh, t)|$ is sufficiently small in some interval of time?

Problem 2. For a given fixed continuous model and a chain of given form, is it possible to choose the number of points and other numerical parameters of the chain to give a sufficiently small quantity $|y_j(t) - u(jh, t)|$ in some interval of time?

The question posed in Problem 2 was answered in the affirmative in the case of the wave equation by D. Bernoulli and Lagrange [14, 15].

Our reason for agreeing with this is that if, formally, we take the limit as $N \to \infty, h \to 0$ in the problem for a chain (2.3), (2.4), (2.5) with $R_1 = R_2 = 0, L_0 = 1, \xi = 0$ under conditions (3.1), we obtain the solution of the corresponding the problem (1.3)–(1.5).

A rigorous justification for taking the limit can be obtained from a proof that the method of straight lines converges in the case of the wave equation. A proof for any fixed time interval is given in [7, 8], for example.

In this connection, in a number of publications of an applied nature it is assumed that the solutions of problems (2.3), (2.4), (2.5) and (1.1), (1.4), (1.5) can be regarded as close if the number N is large enough. This approach is often used when the solutions of problems for a discrete chain are investigated by replacing it by the corresponding continuous medium. It is essentially this method that is used, in particular, by Zhukovskii [9] (see also [19]).

However, it was shown in [17, 18] that the solutions of the problems for a chain and the corresponding continuous medium in an unbounded time interval might differ, and the larger the number of points

N, the greater the difference could be (in any case if it is a prime number or a power of two), since the chain might experience the so-called surge effect.

An obvious question is how to construct a more exact equation than the wave equation which would describe the surge effect in particular. We shall show that (4.2) can be used for that purpose. To determine the form of the corresponding boundary conditions, we note that for $\xi = 0$ Eq. (4.2) is the Euler–Poisson equation for the following functional, which plays the role of an effect (in the notation of (3.1))

$$\int_0^t\int_0^l\left(\frac{1}{2}\rho u_t^2 + uf + E\sum_{n=1}^{M}\frac{(-1)^n h^{2n-2}}{(2n)!}\left(\frac{\partial^n}{\partial x^n}u\right)^2\right)dxdt$$

with boundary conditions

$$\frac{\partial^{2n}}{\partial X^{2n}}u(0,t) = \frac{\partial^{2n}}{\partial X^{2n}}u(l,t) = 0,\ \ n = 0,\dots,M \tag{4.3}$$

A question which naturally rises is whether problem (4.2), (4.3), (1.7) is well posed. To widen the range of applications, we will examine this question in a slightly more general form.

In the rectangle $\Pi(T_0) = \{(x,t) \mid 0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T_0\}$, where $l > 0$, $T_0 > 0$ are certain constants, consider the equation

$$\sum_{r=0}^{R}\Gamma_r\frac{\partial^r}{\partial t^r}u = 2\sum_{r=0}^{P}H_r\frac{\partial^r}{\partial t^r}\sum_{n=0}^{M}\frac{2h^{2n}}{(2n)!}\frac{\partial^{2n}}{\partial X^{2n}}u(x,t) + hg(x,t),\ \ h = \frac{l}{N} \tag{4.4}$$

with initial conditions (1.7) and boundary conditions (4.3). Here $\Gamma_r (r = 0,\dots,R)$, $H_r\ (r = 0,\dots,P)$, $l > 0$, $N \in \mathbb{N}$ are given constants and $(x,t) \to g(x,t)$, $x \to \beta_r(x)$ $(r = 0,\dots,R-1)$ are given functions. Clearly, without loss of generality, it can be assumed that $\Gamma_R > 0$. Suppose that $R > P$. Putting $u(x,t) = \theta(t)\psi(x)$, we can establish that the eigenfunctions $x \to \psi_k(x)$ and eigenvalues λ_k $(k = 1,\dots)$ of the corresponding homogeneous problem have the form

$$\psi_k(x) = \sin\frac{\pi kx}{l},\ \ \lambda_k = -\sum_{n=1}^{M}\frac{(-1)^n}{(2n)!}\left(\frac{\pi kh}{l}\right)^{2n},\ \ k = 1,\dots \tag{4.5}$$

Figure 1 shows how the eigenvalues μ_k (that is, the eigenvalues λ_k as $M \to \infty$, the dashed curve) and the eigenvalues λ_k depend on k for $M = 1$ (the wave equation), $M = 2q$ and $M = 2q + 1$, where q is a sufficiently large natural number.

It follows from (4.5) that all $\lambda_k > 0$ $(k = 1,\dots)$ only if M is an odd number. In the simplest special case, where $R = 2$, $P = 0$, $\Gamma_0 = \Gamma_1 = 0$, $\Gamma_2 = m$, $H_0 = c$, $G_j(t) = 0$ $(j = 1,\dots,N-1)$, that is, where Eq. (4.4) corresponds to the intermediate continuous model for the classical problem of a vibrating beaded thread, $\lambda_k > 0$ $(k = 1,\dots)$ is a necessary condition for (4.3), (4.4), (1.7) to be a well-posed

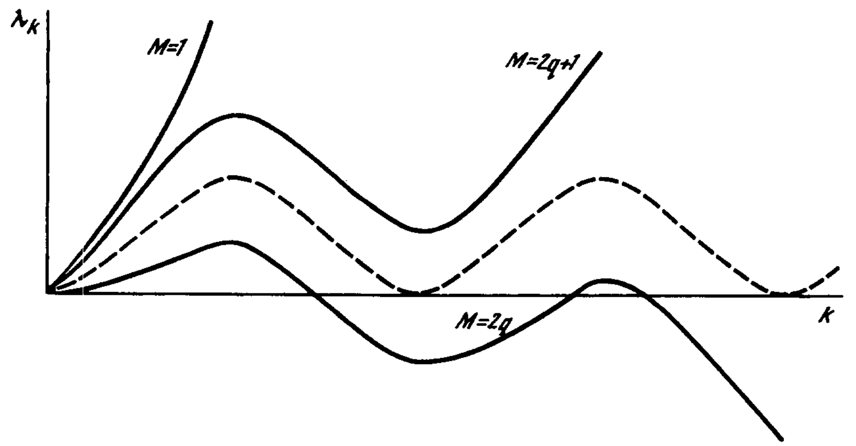

Fig. 1.

problem, since it ensures that none of the functions $\theta_k(t;\lambda_k)$ increases exponentially (as $|\lambda_k| \to +\infty$) in solutions of the form $u_k(x,t) = \theta_k(t;\lambda_k)\psi_k(x)$. In the general case, however, some additional assumptions are required to ensure that the problem is well posed.

Let $\varphi_0(t;\Lambda), \ldots, \varphi_{R-1}(t;\Lambda)$ be a fundamental system of solutions of the equation

$$\sum_{r=0}^{R}\Gamma_r\frac{d^r}{dt^r}\varphi+\Lambda\sum_{r=0}^{P}H_r\frac{d^r}{dt^r}\varphi=0 \tag{4.6}$$

with the functions $t \to \varphi_s(t;\Lambda)$ chosen so that

$$\frac{d^r}{dt^r}\varphi_s(0;\lambda)=\delta_s^r,\quad r=0,\ldots,R-1;\quad s=0,\ldots,R-1 \tag{4.7}$$

where δ_s^r is the Kronecker delta. Suppose that there are constants $D > 0$, $\nu > 0$ and a smooth function $t \to \Omega(t) \geqslant 0$, $\dot\Omega(t) \geqslant 0$, $\Omega(0) = 0$ for which, for each solution $\varphi_s(t;\Lambda)$, $(s = 0, \ldots, R-1)$, the condition

$$\left|\frac{d^r}{dt^r}\varphi_s(t,\lambda)\right| \leqslant D(\Lambda^{r\nu}+1)\dot\Omega(t)<\infty,\quad r=0,\ldots,R-1 \tag{4.8}$$

is satisfied.

Suppose, for example, that Eq. (4.6) corresponds to the classical problem of a vibrating beaded thread. Then $\varphi_0(t;\Lambda) = \cos(t\sqrt{(c\Lambda/m)})$, $\varphi_1(t;\Lambda) = \sin(t\sqrt{(c\Lambda/m)})/\sqrt{(c\Lambda/m)}$, so that we can put $D = \max\{1, c/m\}$, $\nu = 1$, $\Omega(t) = t^2/2$.

Finally, to ensure that problem (4.3), (4.4), (1.7) is well posed (if the solution is understood in the classical sense) we make the following assumptions: $\beta_r \in C^q[0,l]$, $g \in C^q[0,l]$

$$\begin{gathered}\frac{\partial^{2n}}{\partial x^{2n}}\beta_r(0)=\frac{\partial^{2n}}{\partial x^{2n}}\beta_r(l)=0,\quad \frac{\partial^{2n}}{\partial x^{2n}}g(0,t)=\frac{\partial^{2n}}{\partial x^{2n}}g(l,t)=0\\ r=0,\ldots,R-1;\quad n=0,\ldots,MR\nu;\quad q\geqslant 2MR\nu+1\end{gathered} \tag{4.9}$$

It can be shown by standard arguments that, under the above assumptions, problem (4.3), (4.4), (1.7) is well posed.

5. APPROXIMATION THEOREMS

The first approximation theorem. Assuming, as before, that $\Gamma_R > 0$, $R > P$, we consider the following problem for a chain

$$\sum_{r=0}^{R}\Gamma_r\frac{d^r}{dt^r}y_j=\sum_{r=0}^{P}H_r\frac{d^r}{dt^r}(y_{j+1}-2y_j-y_{j-1})+G_j(t) \tag{5.1}$$

with boundary and initial conditions (2.4) and (2.5).

The corresponding problem for the intermediate equation has the form (4.3), (4.4), (1.7) with $H_r = B_r$ $(r = 0, \ldots, P)$. Suppose that the conditions under which this problem is well posed are satisfied.

Let

$$g(x,t)=\frac{2}{Nh}\sum_{j=1}^{N-1}G_j(t)D_{N-1}^*(x,j),\quad \beta_r(x)=\frac{2}{N}\sum_{j=1}^{N-1}\alpha_r(j)D_{N-1}^*(x,j),\quad r=0,\ldots,R-1$$

$$D_{N-1}^*(x,j)=D_{N-1}\left(\frac{\pi}{N}\left(j-\frac{x}{h}\right)\right)-D_{N-1}\left(\frac{\pi}{N}\left(j+\frac{x}{h}\right)\right),\quad D_{N-1}(a)=\frac{\sin(a(N-\frac{1}{2}))}{2\sin(a/2)}$$

($D_{N-1}(\cdot)$ is the Dirichlet kernel).

Then for any $0 \leqslant t \leqslant T_0$ $(T_0 > 0)$

$$\left|y_j(t) - u(jh,t)\right| \leq K_1(\Omega(T_0)\exp(5C_2T_0) + \int_0^{T_0}\Omega(t)\exp(5C_2t)dt)\frac{\pi^{2M+2}}{(2M+2)!} \tag{5.2}$$

$$K_1 = 2NC_1 \max\left\{R\max_{j,r}|\alpha_r(j)|,\ \sup_{j,t\leq T_0}\left|G_j(t)\right|\right\}((4+\pi^4/12)^{Rv}+1)$$

$$C_1 = DP\max_{0\leq r\leq P}|H_r|/\Gamma_R,\quad C_2 = R\max\left\{\max_{0\leq r\leq R-1}|\Gamma_R|,\ \max_{0\leq r\leq P}|H_r|\right\}/\Gamma_R$$

The proof of this theorem is given in the Appendix.

Corollary. Clearly, since

$$\lim_{M\to\infty}\pi^{2M+2}/(2M+2)! = 0$$

by choosing a sufficiently large odd number M for a fixed value of N the quantity $|y_j(t) - u(jh, t)|$ can be made as small as desired in any finite interval $[0, T_0]$. Thus, Theorem 1 provides a solution of Problem 1.

The second approximation theorem. Now consider the problem for the intermediate equation (4.4) with $\Gamma_r = h\gamma_r$, $r = 0, \ldots, R$, $H_r = b_r/h$, $r = 0, \ldots, P$. Problem (4.3), (4.4), (1.7) can be regarded as an intermediate continuous model for the chain (5.1), (2.4), (2.5). The quantities γ_r ($r = 0, \ldots, R$), b_r ($r = 0, \ldots, P$), $l > 0$, number $M \geq 1$, and functions $x \to \beta_r(x)$ ($r = 0, \ldots, R-1$), $(x, t) \to g(x, t)$ which characterize the continuous model will be taken as given. For each $N \in \mathbb{N}$ we put $h = h(N) = l/N$.

Suppose that the conditions for problem (4.3), (4.4), (1.7) to be well posed are satisfied for each h. Let

$$G_j(t) = \frac{h}{l}\int_0^l g(x,t)D^*_{N-1}(x,j)dx,\quad \alpha_r\{j\} = \frac{1}{l}\int_0^l \beta_r(x)D^*_{N-1}(x,j)dx$$

$$r = 0,\ldots,R-1$$

Then for any $N_0 \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $N \geq N_0$, we have

$$\left|y_j(t) - u(jh,t)\right| \leq K_6\Omega_1(T_0)(N_0^{2M+3}N^{-2M}\exp(K_3N_0T_0) + N_0^{-1}) \tag{5.3}$$

$$K_6 = \max\{2K_4, K_5\},\quad K_5 = 2K_3\left(\max_{r,x}|\beta_r(x)| + \sup_{x,t\leq T_0}|g(x,t)|\right)$$

$$K_2 = 2Dl^2\left(R\max_{r,x}\left|\beta_r''(x)\right| + \sup_{x,t\leq T_0}|g_{xx}(x,t)|/\gamma_R\right)/\pi^2$$

$$K_4 = C_2\frac{\pi^2}{4l^2},\quad K_3 = C_1\frac{\pi^{2M+2}}{l^2(2M+2)!}\left(\left(4+\frac{\pi^2}{12}\right)^{Rv}+1\right)$$

$$\Omega_1(T_0) = \sup_{t\leq T_0}\left(\dot{\Omega}(t) + \Omega(t) + \Omega(t)\exp(C_2t) + \int_0^t\Omega(t)\exp(C_2t)dt\right)$$

The proof of this theorem is given in the Appendix.

Corollary. For a number N_0 so chosen that $K_6\Omega_1(T_0)N_0^{-1}$ is sufficiently small, we choose a value of N for which

$$N \geq N_0^{2M+3}\exp(aK_3N_0T_0),\quad a > 1$$

Then the right-hand side of the bound (5.3) can be made as small as required by choosing sufficiently large N_0 and N of the given form. Thus, Theorem 2 gives a solution of Problem 2.

6. SYSTEMS WITH NON-LINEAR RELATIONS AND NON-LINEAR CREEP AND RELAXATION

Suppose that the basic equation of motion has the form (2.1), but instead of the linear constitutive relation (2.2) consider, for example, the non-linear relation

$$\sum_{r=0}^{R_1} L_r \frac{d^r}{dt^r} s_j = \sum_{r=0}^{R_2} B_r \left|\frac{d^r}{dt^r}\Delta_j\right|^p \operatorname{sgn}\left(\frac{d^r}{dt^r}\Delta_j\right), \quad p>0 \tag{6.1}$$

a natural extension of relation (3.4) to the case $c_1 = 0$. Eliminating s_j from (2.1) and (6.1), we obtain an equation for y_j

$$\sum_{r=0}^{R_1} L_r \frac{d^r}{dt^r}(m\ddot{y}_j + \xi\dot{y}_j - F_j) = \sum_{r=0}^{R_2} B_r \left(\left|\frac{d^r}{dt^r}\Delta_{j+1}\right|^p \operatorname{sgn}\left(\frac{d^r}{dt^r}\Delta_{j+1}\right) - \left|\frac{d^r}{dt^r}\Delta_j\right|^p \operatorname{sgn}\left(\frac{d^r}{dt^r}\Delta_j\right)\right) \tag{6.2}$$

Remark. There are several ways of extending the constitutive relations of the linear theory of viscoelasticity to the non-linear case. It can be shown that relation (6.1) is covered by the non-linear genetic theory of Liderman and Rozovskii (see [3, 20, 21], for example).

If now, for a continuous medium (1.1), we write the constitutive relation

$$\sum_{r=0}^{R_1} L_r \frac{\partial^r}{\partial t^r}\sigma = \sum_{r=0}^{R_2} b_r \left|\frac{\partial^r}{\partial t^r}\varepsilon\right|^p \operatorname{sgn}\left(\frac{\partial^r}{\partial t^r}\varepsilon\right) \tag{6.3}$$

analogous to (6.1), then by eliminating the function σ from (1.1) and (6.3) we obtain the equation

$$\sum_{r=0}^{R_1} L_r \frac{\partial^r}{\partial t^r}(\rho u_{tt} + \eta u_t - f) = \sum_{r=0}^{R_2} b_r \left|\frac{\partial^r}{\partial r^r} u_x\right|^p \operatorname{sgn}\left(\frac{\partial^r}{\partial t^r} u_x\right) \tag{6.4}$$

Thus in this case the intermediate equation of order $(p + 1)$ for a chain is the same as the corresponding equation for a strongly non-linear continuous medium if $\rho = m/h$, $\eta = \xi/h$, $l = Nh$, $b_r = B_r h^p$.

It is possible to construct series of exact solutions of the form $u(x, t) = \psi(x)\theta(t)$ for Eq. (6.4), as shown in [20]. Of course, it is difficult to obtain rigorous assertions analogous to Theorems 1 and 2 in the non-linear case. Replacing a chain (6.2) by Eq. (6.4) is normally justified by means of computer experiments. The same method can be used to demonstrate that this is legitimate in the case here.

It should be emphasized that it is because exact solutions for Eq. (6.4) exist [21] that the solutions can be compared in this way. If Eq. (6.4) had to be solved by a grid method, then rather than the solution of Eq. (6.4), it would be the solution of the corresponding discrete chain with respect to x and t that was actually sought.

It might prove to be very difficult to conduct an analytic investigation of the solutions of a non-linear discrete chain (the Fermi–Pasta–Ulam problem is a good example [13]). This is why an attempt is often made to obtain a continuous analogue of the original chain which is easier to analyse. In particular, the Boussinesq equation [1] can be used for the Fermi–Pasta–Ulam problem.

One reason why the continuous analogue is often easier to analyse that the original chain is made clear by the following example.

Suppose that the basic equation of motion has the form (2.1), and the constitutive relation is taken in the form (3.4), where $c_1 = 0, p > 0$. After eliminating s_j, we obtain the following equation for the displacements $y_j(t)$

$$m\ddot{y}_j = B(|y_{j+1} - y_j|^p \operatorname{sgn}(y_{j+1} - y_j) - |y_j - y_{j-1}|^p \operatorname{sgn}(y_j - y_{j-1})) \tag{6.5}$$

Suppose that the problem has fixed boundaries, that is, $y_0(t) = y_N(t) = 0$. Equation (6.5) has exact solutions which are of the form

$$y_j(t) = I(j)Q(t) \tag{6.6}$$

where the functions I and Q will be solutions of the following problems

$$|I(j+1)-I(j)|^p \operatorname{sgn}(I(j+1)-I(j))-$$
$$-|I(j)-I(j-1)|^p \operatorname{sgn}(I(j)-I(j-1))+\mu I(j)=0,\quad I(0)=I(N)=0 \tag{6.7}$$

$$m\ddot{Q}(t)+\mu b|Q(t)|^p \operatorname{sgn} Q(t)=0 \tag{6.8}$$

The parameter μ here plays the part of an eigenvalue.

We will now consider the continuous analogue of this chain in the form

$$\rho u_{tt}=b\frac{\partial}{\partial x}(|u_x|^p \operatorname{sgn}(u_x)),\quad \rho=\frac{m}{h},\quad b=Bh^p,\quad l=Nh \tag{6.9}$$

where h is the distance between the physical points in static equilibrium. Equation (6.9) has the exact solutions of the form $u(x, t) = \psi(x)\theta(t)$, where the functions ψ and θ will be the solutions of the following problems

$$(|\psi'(x)|^p \operatorname{sgn}(\psi'(x))'+\lambda\psi(x)=0,\quad \psi(0)=\psi(l)=0 \tag{6.10}$$

$$\rho\ddot{\theta}(t)+\lambda b|\theta(t)|^p \operatorname{sgn}(\theta(t))=0 \tag{6.11}$$

where the parameter λ plays the part of an eigenvalue.

Although Eqs (6.8) and (6.11) are similar, the eigenvalue problem is much more difficult for the non-linear difference equation (6.7) than for the non-linear differential equation (6.10) because, generally speaking, solutions of non-linear difference equations, even first-order, behave in a much more complex way than those of the analogous differential equations (see [22], for example).

It might therefore be easier to make an analytic investigation of the solution of the continuous analogue of the chain than of the chain itself.

7. APPENDIX

Lemma. Let $\varphi_s(t, \Lambda)$ $(s = 0, \ldots, R-1)$ be a fundamental solution of Eq. (4.7). Then Λ_1, Λ_2 satisfy the inequality

$$|\varphi_s(t;\Lambda_1)-\varphi_s(t;\Lambda_2)|\leq C_1|\Lambda_1-\Lambda_2|(\Lambda_1^{R\nu}+1)\Omega(t)\exp(C_2(\Lambda_2+1)t)$$

Proof. By reducing Eq. (4.7) to a system and using Gronwall's lemma, we obtain the required inequality.

Outline of the proof of Approximation Theorem 1. To simplify the calculations, we will restrict ourselves to the case where

$$\alpha_r(j)=0,\ r=0,\ldots,R-1;\ j=1,\ldots,N-1$$

The solution of Eq. (5.1) has the form

$$y_j(t)=\sum_{k=1}^{N-1}Q_k(t)I_k(j);\quad Q_k(t)=\frac{1}{\Gamma_R}\int_0^t\varphi_R(t-\tau;\mu_k)q_k(\tau)d\tau,\quad I_k(j)=\sin\frac{\pi kj}{N} \tag{7.1}$$
$$q_k(t)=\frac{2}{N}\sum_{j=1}^{N-1}F_j(t)I_k(j),\quad \mu_k=4\sin^2\frac{\pi k}{2N};\quad k=1,\ldots,N-1$$

The solution of Eq. (4.4) has the form

$$u(x,t)=\sum_{k=1}^{\infty}\theta_k(t)\psi_k(x) \tag{7.2}$$

$$\theta_k(t)=\frac{1}{\gamma_{R_0}}\int_0^t\varphi_R(t-\tau;\lambda_k)p_k(\tau)d\tau,\quad p_k(t)=\frac{2}{l}\int_0^l g(x,t)\psi_k(x)dx,\quad k=1,\ldots \tag{7.3}$$

where $\psi_k(x)$, λ_k have the form (4.5).

By virtue of the construction of the function $(x, t)\to g(x, t)$ we have: $p_k(t) = q_k(t)$ $(k = 1, \ldots, N-1)$, $p_k(t) = 0$ $(k = N, \ldots)$. Thus

$$|y_j(t)-u(jh,t)|\leq\sum_{k=1}^{N-1}\int_0^t|\varphi_R(t-\tau;\lambda_k)-\varphi_R(t-\tau;\mu_k)||q_k(\tau)|d\tau \tag{7.4}$$

Using the lemma with $\Lambda_1 = \lambda_k$, $\Lambda_2 = \mu_k$, we obtain the required result from (7.4).

Outline of the proof of Approximation Theorem 2. As in the proof of Theorem 1, in order to simplify the calculations we will restrict ourselves to the case where $\beta_r(x) = 0, r = 0, \ldots, R-1: 0 \leqslant x \leqslant l$.

The solution of Eq. (5.1) has the form (7.1), and the solution of Eq. (4.4) has the form (7.2). In this case, however, the functions $t \to \theta_k(t)$ defined by formula (7.3) are generally speaking no longer necessarily equal to zero for $k \geqslant N$. Nevertheless, by virtue of the construction of the functions $t \to G_j(t)$, we have: $q_k(t) = p_k(t)$ $(k = 1, \ldots, N-1)$. Using conditions (4.9), we obtain the bounds

$$\sup_{t \leqslant T_0} \sum_{k=N_0+1}^{N-1} |q_k(t)||I_k(j)| \leqslant \sup_{t \leqslant T_0} \sum_{k=N_0+1}^{\infty} |p_k(t)||I_k(j)| \leqslant$$

$$\leqslant 2\left(\frac{l}{\pi}\right)^2 \sup_{X;t \leqslant T_0} |g''_{xx}(x,t)| \frac{1}{N_0} \int_0^t \Omega(t)dt \tag{7.5}$$

Thus, since

$$|y_j(t) - u(jh,t)| \leqslant \sum_{k=1}^{N_0} |Q_k(t) - \theta_k(t)||\psi_k(jh)| +$$

$$+ \sum_{k=N_0+1}^{N-1} |Q_k(t)||I_k(j)| + \sum_{k=N_0+1}^{\infty} |\theta_k(t)||\psi_k(jh)|$$

using the lemma with $\Lambda_1 = \lambda_k/h^2$, $\Lambda_2 = \mu_k/h^2$ and bound (7.5), we obtain the required result.

Remark. The difference operator $(y_{j+1} - 2y_j + y_{j-1})$ on the right-hand side of Eq. (5.1) can be replaced by the more general operator

$$\sum_{q=-R_3}^{R_4} a_q y_{j+q}, \quad R_3 \geqslant 0, \quad R_4 > 0$$

for certain coefficients a_q. The proof of the theorems follow the same scheme as before if the existing eigenfunctions $x \to \psi_k(x)$ $(k = 1, \ldots)$ and $j \to I_k(j)$ $(k = 1, \ldots, N-1)$ here form the orthogonal systems, with y_k forming a complete system, and $\psi_k(jh) = I_k(j)$. But without explicit expressions for the eigenfunctions ψ_k, I_k we do not yet have effective criteria for testing the validity of these additional conditions.

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REFERENCES

1. BLAND, D. R., *The Theory of Linear Viscoelasticity*. Pergamon Press, Oxford, 1960.
2. REINER, M., *Rheology*. Springer, Berlin, 1958.
3. PAL'MOV, V. A., *Vibrations of Elastoplastic Bodies*. Nauka, Moscow, 1976.
4. ISHLINSKII, A. Yu., Longitudinal vibrations of a rod with a linear law of aftereffect and relaxation. *Prikl. Mat. Mekh.*, 1940, **4**, 79–92.
5. BELLMAN, R. and COOKE, K. L., *Differential-difference Equations*. Academic Press, New York, 1963.
6. LACROIX, S. F., *Traité du Calcul Différentiel et Calcul Intégral*. Bachelies, Paris, Vol. 3.
7. LEBEDEV, V. I., Equations and convergence of the differential-difference (straight-line) method. *Vestnik MGU. Ser. Fiz.-Mat. i Yestest. Nauk*, 1955, 10, 47–57.
8. BUDAK, B. M., On the method of straight lines for certain boundary-value problems. *Vestnik MGU. Ser. Mat., Mekh., Astron., Fiz., Khimii*, 1956, **1**, 3–12.
9. ZHUKOSKII, N. Ye., The work of continuous and non-continuous traction devices in pulling a train from its position at the beginning of its motion. In *Complete Collected Papers* Vol. 3, 647–679. Gostekhizdat, Moscow and Leningrad, 1949.
10. BEREZIN, I. S. and ZHIDKOV, N. P., *Computational Methods*. Fizmatgiz, Moscow, 1962, Vol. 2.
11. KRUSKAL M., Asymptology in numerical computations: Progress and plans on the Fermi–Pasta–Ulam problem. *Proc. IBM Scientific Comp. Symp. on Large-Space Problem Division*. New York, 1963, pp. 43–66.
12. ZABUSKY N., Exact solution for the vibrations of nonlinear continuous model string. *J. Math. Phys.*, 1962, **3**, 1028–1039.
13. FERMI E., PASTA J. and ULAM S., Studies of nonlinear problems. In *Collected Works of E. Fermi*. Chicago University Press, Chicago, 1965, Vol. 2, pp. 978–988.
14. BERNOULLI D., Reflexions et éclaircissements sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie. In *Mémoires de l'Académie Roy. de Berlin*, 1973, Vol. 9, pp. 147–172.
15. LA GRANGE, J. L., *Mechanique Analitique*. Desaint, Paris, 1788.
16. STRUTT, J. W. (LORD RAYLEIGH), *The Theory of Sound*. Macmillan, London, 1926 ,Vol. 1.
17. KURHCANOV, P. F., MYSHKIS, A. D. and FILIMONOV, A. M., Vibrations of rolling stock and a theorem of Kronecker. *Prikl. Mat. Mekh.*, 1991, **55**, 989–995.

18. FILIMONOV, A. M., KURCHANOV, P. E. and MYSHKIS, A. D., Some unexpected results in the classical problem of vibrations of a string with *n* beads when *n* is large. *C. R. Acad. Sci. Ser.*, 1991, 1, **313**, 13, 961–965.
19. LUR'YE, A. I., *Operational Calculus and its Applications to Mechanics*. Gostekhizdat, Moscow, 1950.
20. LOKSHIN, A. A. and SAGOMONYAN, Ye. A., *Non-linear Waves in Solid State Mechanics*. Izd. MGU, Moscow, 1989.
21. FILIMNOV, A. M., Exact solutions of some problems of the vibrations of a one-dimensional continuous medium with non-linear inheritance and features of the corresponding non-linear eigenvalue problems. *Izv. Ross. Akad. Nauk. MTT.*, 1992, **27**, 6, 121–128.
22. SHARKOVSKII, A. M., MAISTRENKO, Yu. L. and ROMANENKO, Ye. Yu., *Difference Equations and their Applications*. Naukova Dumka, Kiev, 1986.

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